Topological Bulk Discretization

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Physics-Free Problem Statement

In purely graph theoretic terms (ignoring physics motivations), the **Discrete Bulk Reconstruction Problem** (Discrete BuRP) is as follows:

> Given an ordered set of n vertices V_{bdry} and a vector $S \in \mathbb{R}^{2^n}$, construct a weighted graph $G = (V, E)$ such that $V_{bdry} \subseteq V$, and for all $R \subseteq V_{bdry}$, the min-cut separating R from $\bar{R}$ has value S_R.

Note $S_R = S_{\bar{R}}$ for all R, and $S_\emptyset = S_{V_{bdry}} = 0$, so really $S \in \mathbb{R}^{2^{n-1}-1}$.

The Discrete BuRP is essentially the inverse of the min-cut problem. Instead of being given a graph and being asked to find its min-cuts, we are given a list of min-cuts and are asked to find a consistent graph.

Background and Motivation

One of the leading models of quantum gravity is the **Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence[1]**, which uses the holographic principle to draw a connection between a theory of gravity (AdS, also called "the bulk") and a theory of quantum mechanics (CFT, also called "the boundary"). Though this model describes a universe we probably do not live in, it could give us insights into our own universe.

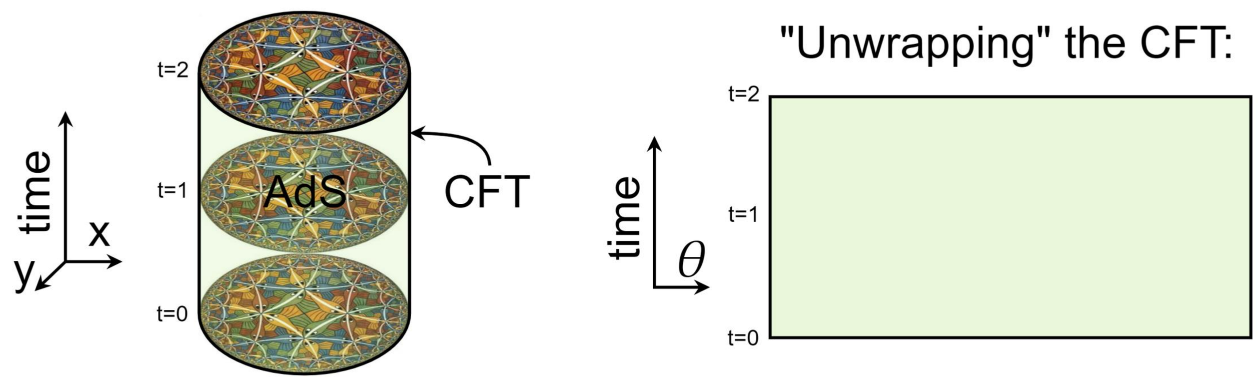

Figure 1. A (2+1)-dimensional AdS bulk and its (1+1)-dimensional CFT boundary

Both physicists and computer scientists are interested in knowing how difficult it is to reconstruct the geometry of the gravitational bulk given the state on the quantum mechanical boundary. This is known as the **Bulk Reconstruction Problem.** If solved, this would give us one possible way the geometry of spacetime might arise from quantum information.

Abstract

This work explores a problem within a simplification of this model, namely the Discrete Bulk Reconstruction Problem (Discrete BuRP). We are given the entanglement structure of a discrete CFT boundary, and we are asked to reconstruct the discrete geometry of the AdS bulk. More concretely, we are given a list of min-cut values, and we are asked to construct a graph with that specific min-cut structure. Recent results from Aaronson and Pollack explored instances of the Discrete BuRP which are efficient to solve, particularly contiguous entropy vectors[2].

In this work, we show how these bulk graphs can be thought of as coarse-grainings of continuous 2D AdS spaces. We also explore how modifying various topological properties of the AdS—like the connectedness, number of boundaries, or placement of boundary regions—can simplify the corresponding bulk graphs. Finally, we make some remarks on the runtimes of algorithms that determine key properties of bulk graphs.

Physical Interpretation

In the problem statement above, R represents some subsystem of an n-partite quantum state existing on the boundary. S_R represents the entanglement entropy of that subsystem, $S(R)$. Min-cuts are meant to represent extremal surfaces (geodesics) in the bulk.

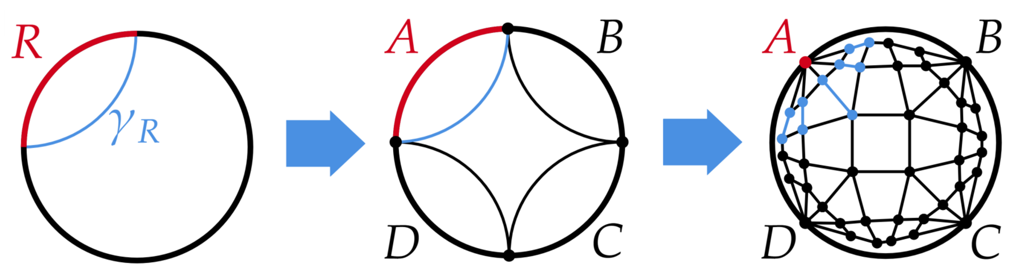

Figure 2. Discretizing the boundary into n regions, then discretizing the bulk into a graph.

Prior Work

Aaronson and Pollack[2] entirely solved the case of 2D trivial bulk topology (no wormholes). In this case, the exponentially long input only has $\binom{n}{2}$ degrees of freedom. They give three successively better algorithms for this case: the bulkless **(Figure 3.1)**, chord **(Figure 3.2)**, and diamondwork **(Figure 3.3)** constructions.

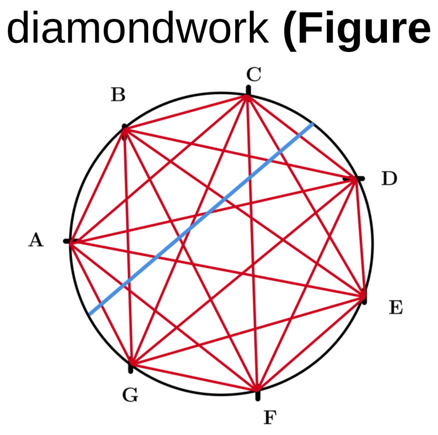

Figure 3.1

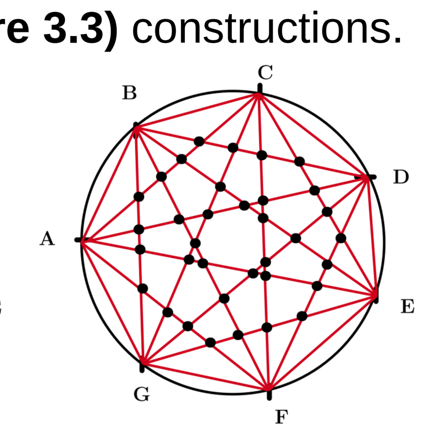

Figure 3.2

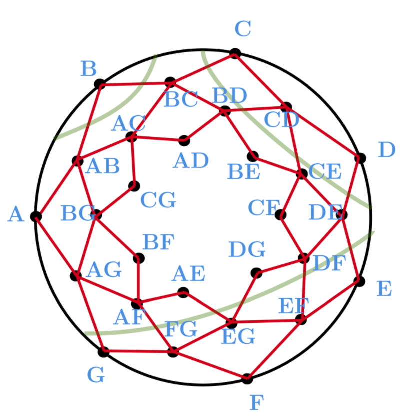

Figure 3.3

Images taken with permission from [Aaronson and Pollack 2022][2].

Results and Figures

Issue: Non-planar graphs can be non-physical. **(Figure 4.1)**

- The Aaronson-Pollack approach: ignore entropy vectors that require non-trivial topology and introduce bulk vertices when necessary. **(Figure 4.2)**
- Our approach: embed our graphs in higher genus surfaces. **(Figure 4.3)**

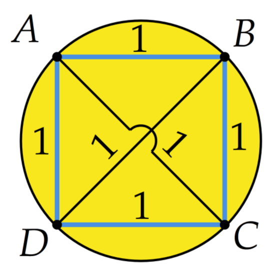

Figure 4.1

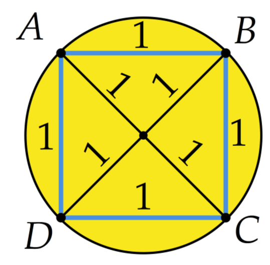

Figure 4.2

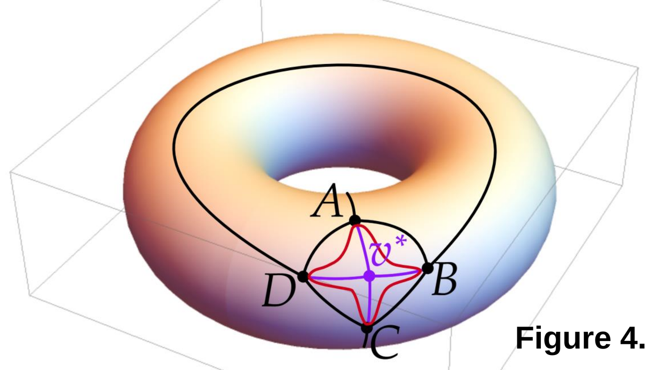

Figure 4.3

> **Theorem 1**
> G bulk embeds on S iff G^* embeds on S^*.

Here S is a surface with k punctures, and G is a graph with n boundary vertices partitioned into k boundaries. G^* is the graph G after introducing one new node for each boundary connected to every vertex on that boundary. S^* is the closed surface after gluing a disk to the edge of each puncture in S. Both directions can be proved pictorially. The reverse direction uses ribbon graphs to represent the graph embedding of G^*.

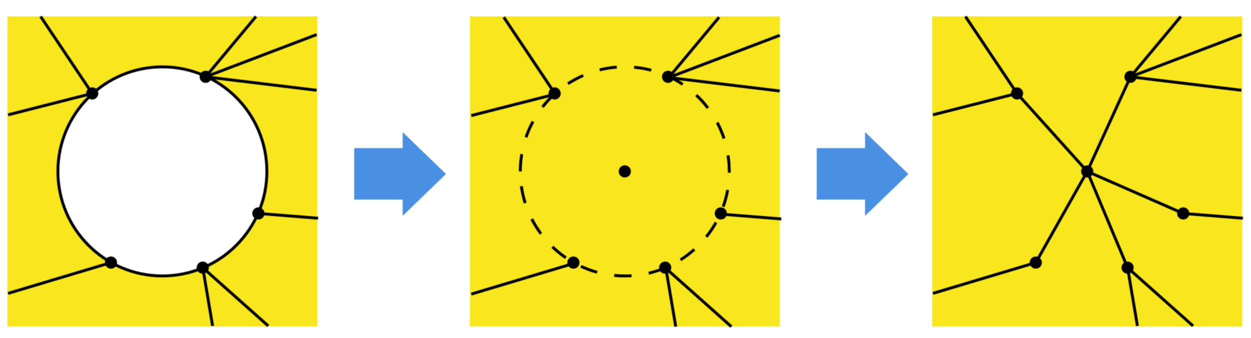

Figure 5.1. Converting S to S^*, then G to G^*.

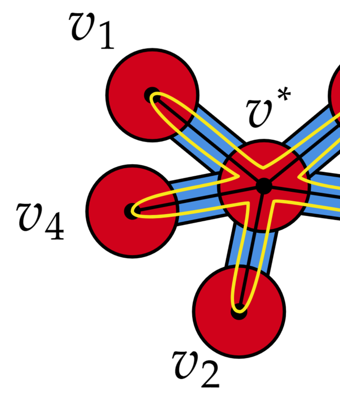

Figure 5.2. The reverse.

> **Theorem 2**
> Embeddings imply physically meaningful (spatial) geodesics.

Physically reasonable geodesics should:

- be closed loops or terminate on a boundary.
- pass through the edges they are meant to cut exactly once.

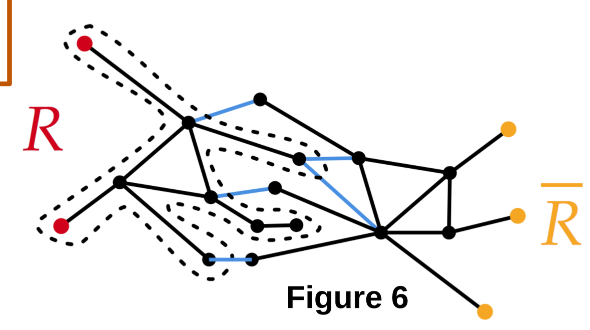

Figure 6

Theorem 1 lets us think of bulk embeddings as graph embeddings on closed surfaces. After embedding a graph on a surface, the boundary of an ϵ-net about the connected component of R after removing all edges in the min-cut will be a closed loop that cuts each edge in the min-cut exactly once. Reintroducing punctures for each boundary may cut these closed loops into geodesics that terminate on the boundaries.

This new picture helps us attempt to solve the general Discrete BuRP with arbitrary entropy vectors. For instance, we find solutions to an entropy vector for which the diamondwork construction fails (Figure 7.1) by permuting boundary regions (Figure 7.2), increasing genus (Figure 7.3), and/or allowing the bulk to be non-orientable (Figure 7.4). We're currently working on improving over a trivial algorithm for arbitrary entropy vectors.

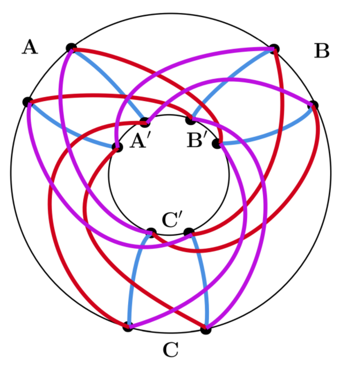

Figure 7.1

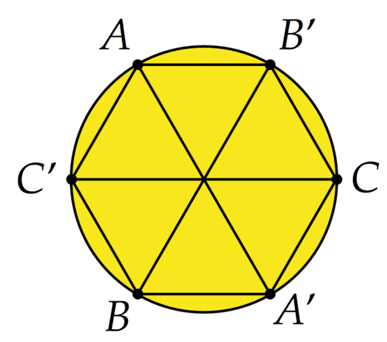

Figure 7.2

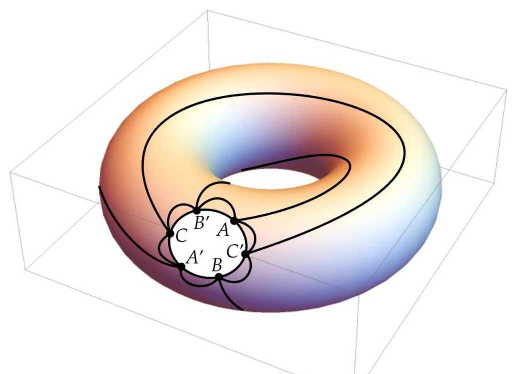

Figure 7.3

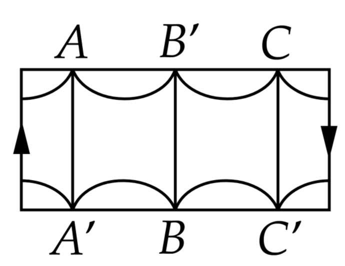

Figure 7.4

Algorithm	Input type	Input size	Runtime	# Verts	# Edges	Embed?	Genus
Bulkless[2]	Contiguous	$O(n^2)$	$O(n^2)$	0	$O(n^2)$	No	0
Chord[2]	Contiguous	$O(n^2)$	$O(n^2)$	$O(n^4)$	$O(n^4)$	Yes	0
Diamondwork[2]	Contiguous	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(n^2)$	Yes	0
High-genus	Contiguous	$O(n^2)$	$O(n^2)$	0	$O(n^2)$	Yes	$O(n^2)$
Naive high-genus[3]	Arbitrary	$2^{n-1}-1$	$\exp(2^{2^n})$	$O(2^{2^n})$	$O(4^{2^n})$	Yes	$O(4^{2^n})$

Conclusions

The goal of this work is to solve the Discrete Bulk Reconstruction Problem for non-trivial topologies. We reframe the problem in terms of topological graph theory, the study of embedding graphs on surfaces, ensuring our solutions are physically meaningful. We see a tradeoff between surface genus and the number of bulk vertices, and we gain insights that might help us in the search for more efficient algorithms.

This line of research is just getting started, and eventually we'd like to make the problem even more physically relevant by including a dimension of time, increasing the dimension of the bulk, and generalizing the problem to simplicial complexes instead of graphs.

References

[1] Juan M. Maldacena. The Large N Limit of Superconformal Field Theories and Supergravity, 1997, *Adv. Theor. Math. Phys.* 2:231-252,1998; arXiv:hep-th/9711200. DOI: 10.1023/A:1026654312961.

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[3] Ning Bao, Sepehr Nezami, Hirosi Ooguri, Bogdan Stoica, James Sully, Michael Walter: "The Holographic Entropy Cone", 2015, JHEP 09 (2015) 130; arXiv:1505.07839. DOI: 10.1007/JHEP09(2015)130.